WHAT NOW: STATICALLY INDETERMINATE?

Statically indeterminate structures generally receive (too) little attention in structural engineering education. This is a pity, as there is much to be learned from them; especially regarding calculations using the finite-element method (FEM). This article explains how this works.

ERIK TEMPELMAN

First, a brief definition: a structure is considered statically determinate if all unknown internal and external forces can be determined using the equations of static equilibrium. The materials from which these structures are made do not matter, provided they possess sufficient stiffness and strength. The situation differs for statically indeterminate structures: static equilibrium alone is insufficient to determine the unknowns. In such cases, we must also consider deformations; consequently, the material of the structure does become a relevant factor.

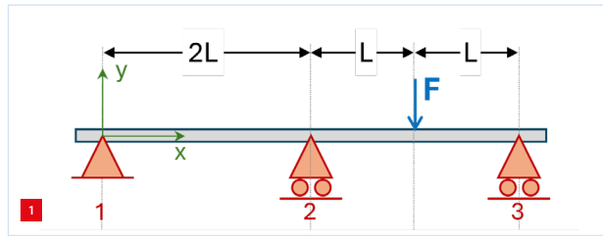
Second, two points to note: (1) strictly speaking, the term 'statically indeterminate' encompasses both statically under-determined and statically over-determined structures – in this article, we refer to the latter; and (2) in some cases, statically indeterminate situations can be analysed using static force equilibrium combined with symmetry considerations. We do not consider this latter scenario in the present article.

Let us look at an example right away: the beam supported at three points (Figure 1). There are four unknowns in total: the horizontal reaction F_{1x} and the three vertical reactions F_{1y}, F_{2y} and F_{3y}. In this 2D scenario, the beam has three degrees of freedom, meaning three independent reactions would have sufficed; the beam is indeed statically indeterminate. There is a single external load; specifically, a downward load F positioned exactly halfway between the second and the third support.

Stiff-beam approach

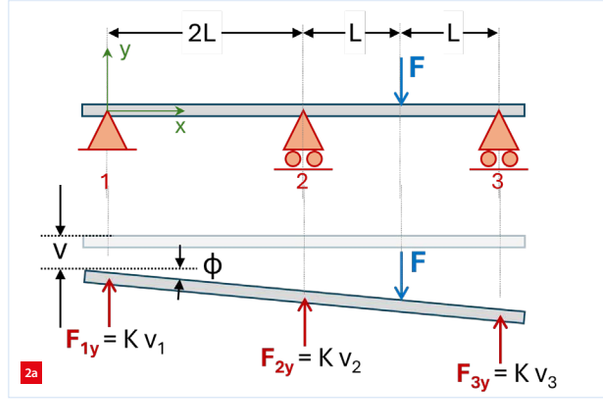
We can take two approaches to analyse this problem. First, we can determine the reactions assuming the supports can deform slightly while the beam itself remains perfectly stiff. Second, we can consider the reverse scenario: a flexible beam on perfectly stiff supports. We begin here with the first method, the 'stiff-beam approach'; see Figure 2a.

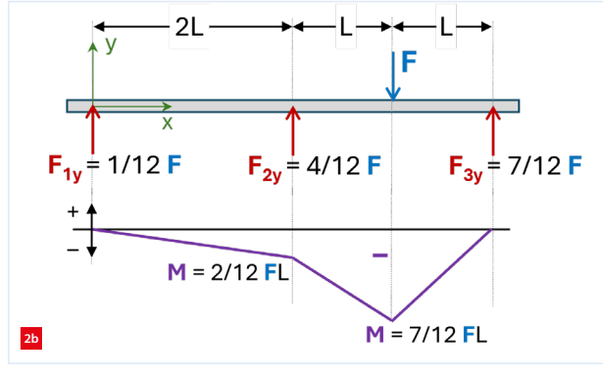
Here, we assume that the three vertical supports deform as if they were springs with a specific spring constant K (in N/m). We also assume that the beam shifts downwards by a distance



Statically indeterminate beam with external load F.

$v_1 = v$ at point 1 and rotates by an angle φ (measured in radians, clockwise). Support 2 moves downwards by a distance $v_2 = v + 2L\sin(\varphi)$; for small values of the angle φ, this simplifies to $v_2 = v + 2L\varphi$. Similarly, we find that $v_3 = v + 4L\varphi$. All this implies that the reaction forces are $F_{1y} = Kv$, $F_{2y} = K(v + 2L\varphi)$, and $F_{3y} = K(v + 4L\varphi)$. After all, at each support, the vertical reaction force must equal the spring constant K multiplied by the downward displacement of that support.





Stiff-beam approach, parts 1 and 2.

AUTHOR'S NOTE
Erik Tempelman is an associate professor in the Faculty of Industrial Design Engineering at Delft University of Technology (NL) and the director of NSFD Engineering Education – NSFD stands for (Not) Safe for Designers. He is also the author of 'NSFD: engineering essays on structures & materials' (Boekengilde, 2020) and a series of articles in *Constructeur* under the NSFD moniker. This article is based on number XXX in that series, which was published in this year's issue 3 of *Constructeur*, which is the Dutch magazine for engineers and mechanical designers. It now appears within the framework of a collaboration between *Constructeur* and *Mikroniek* for the exchange of articles.

www.tudelft.nl/en/ide/about-ide/people/tempelman-e

At first glance, it appears we have made little progress, as we now have three new unknowns: K, v and φ. However, the value of the spring constant K is irrelevant, provided it is not too small. We therefore simply set it to 1. This leaves only v and φ to be determined. Now consider the static equilibrium:

(1) $\Sigma F_x = 0 \quad \rightarrow F_{1x} = 0$ (... as we already knew)
(2) $\Sigma F_y = 0 \quad \rightarrow v + (v + 2\varphi) + (v + 4\varphi) - F = 0$
(3) $\Sigma M_o = 0 \quad \rightarrow 2L(v + 2L\varphi) - 3LF + 4L(v + 4L\varphi) = 0$

The solution is easily found. We use Equation (2) to express v in terms of F and φ, and substitute this expression into Equation (3). Some calculation then yields the three required reactions: $F_{1y} = 1/12 \cdot F$, $F_{2y} = 4/12 \cdot F$, and $F_{3y} = 7/12 \cdot F$. Figure 2b shows these results; the reader is invited to reproduce this analysis.

Also shown in Figure 2b is the bending moment diagram derived from this loading and the corresponding reactions. This is, of course, the actual result we are seeking – specifically the maximum value reached by the internal moment ($M = 7/12 \cdot FL$, at the location of the external load). The latter is directly linked to the maximum material stress via the crucial equation $\sigma = M/W$, where W is the section modulus of the beam cross-section.

Flexible-beam approach

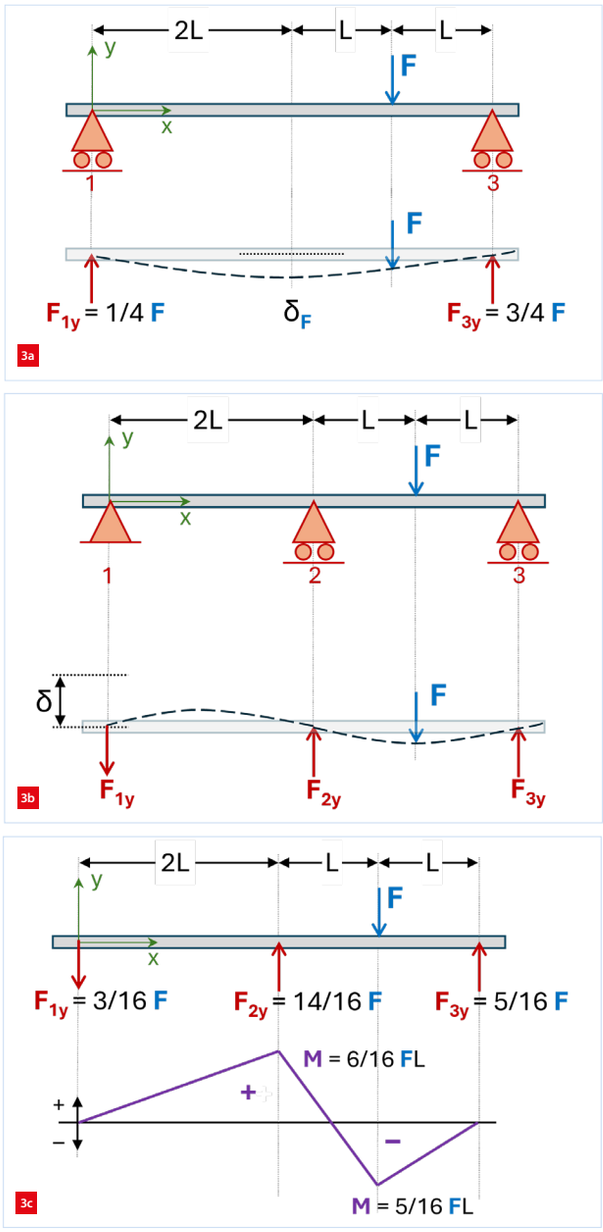
It is time to explore the second approach: that of a flexible beam on stiff supports. Here, we can make good use of the so-called 'forget-me-nots' from beam theory – tools we have encountered several times in this series of articles [1]. We begin by mentally removing the support at point 2 – see Figure 3a. In this scenario, the deflection at the centre – that is, at the location of the unknown reaction force F_{2y} – is equal to $\delta_F = FL^3/48EI \cdot 11/16$.

However, point 2 is constrained and therefore cannot move. This means that the reaction force F_{2y} must deform the beam in such way that point 2 returns exactly to its original position; see Figure 3b. We thus obtain the expression $\delta_{F2y} = F_{2y}L^3/48EI$ for this deflection. This implies that $FL^3/48EI \cdot 11/16 = F_{2y}L^3/48EI$, or in other words, that $F_{2y} = 11/16 \cdot F$.

We determine the remaining two unknowns using static equilibrium. Moment equilibrium about point 1 immediately tells us that $F_{3y} = 13/32 \cdot F$; force equilibrium in the vertical direction then yields $F_{1y} = -3/32 \cdot F$. These results are shown in Figure 3c, which also displays the corresponding bending moment diagram. The maximum moment occurs midway between points 2 and 3 and has a value of $M = 13/32 \cdot FL$.

Stiff beam versus flexible beam

If we compare the two approaches, we observe notable differences. With the stiff-beam approach, the maximum internal bending moment was $M = 7/12 \cdot FL$ – occurring at the point of the external load – whereas with the flexible-beam approach, we arrived at $M = 13/32 \cdot FL$ at point 2. The former



Flexible-beam approach, parts 1, 2 and 3.

is nearly 1.5 times greater than the latter, even though they occur at the same location.

Which of these two different approaches is the correct one? The answer can only be: neither! A real-world beam is never perfectly stiff, nor are its supports. In reality, we will therefore see a combination of both – bearing in mind that deflections are generally far greater than compressive deformations, meaning the flexible-beam approach comes (much) closer to the 'truth' than the stiff-beam approach.

Space frames

So far, we have discussed the way a structure is supported. However, the structure itself may also be statically indeterminate; this is most clearly seen in trusses, i.e., space frames [2]. See Figure 4 for an illustration. Frame (1) has

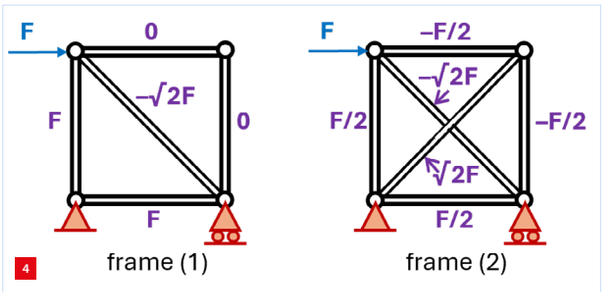
four nodes and five members; exactly the number required to make the frame statically determinate. Frame (2), by contrast, has an extra member, making it statically indeterminate.

In general, a 2D frame with N nodes must have a number of members equal to $2N - 3$ to be statically determinate; in 3D, we use the formula $3N - 6$. With fewer members, the frame is a mechanism ('under-determined'); with more, it is statically indeterminate ('over-determined'). Note, incidentally, that both frames in Figure 4 have statically determinate supports: there are two unknown reaction forces at the bottom left and one at the bottom right.

Figure 4 also shows the internal member forces, indicated in purple. For frame (1), we can determine these forces using static equilibrium. For frame (2), we have imposed an additional condition: the elongation of one diagonal must equal the shortening of the other. Note that this condition arises from symmetry: under the given external load, the frame will deform into a rhombus. And, naturally, the same consideration applies to the other four bars.

At first glance, it might appear that the extra diagonal has made the frame twice as strong; a logical conclusion, considering that the two zero-force members from frame (1) now also carry load in frame (2). In reality, however, this does not happen. The reason is simple: tolerances. Inevitable dimensional variations between the two diagonal members mean that one diagonal will bear more load than the other. This changes the entire scenario, and the factor of two just mentioned will certainly turn out to be lower in practice.

The significance of this effect becomes clear through a simple numerical example. Suppose the members are made of steel, and we assume a maximum material stress of 100 MPa. Hooke's Law tells us that the strain in the members will be a maximum of 0.05% (!). For a length of 1 meter, a member would thus elongate or shorten by only 0.5 mm. This is certainly comparable in magnitude to typical dimensional deviations. Therefore, if we wish to accurately analyse the strength of a truss, it is far better to deal with a statically determinate situation than a statically indeterminate one.



4 *Statically determinate (1) and statically indeterminate (2) truss.*

Implications for FEM calculations

Dear reader, you might be thinking, "Nice calculations, but what am I supposed to do with them?" Let us explain. In structural engineering practice, we typically use the finite-element method (FEM), which handles statically indeterminate situations with ease. However, whether your FEM software actually yields the correct result is another matter entirely. The model analysed by the software has boundaries – specifically at the supports – which we usually treat as infinitely stiff, even though they are not so in reality. In statically indeterminate situations, this can lead to significant discrepancies between the software's output and actual behaviour. Competent validation remains advisable; no matter how impressive the FEM visualisations look, nothing beats a well-executed test on a representative prototype.

There is another issue with statically indeterminate structures, and that concerns tolerances. Look back at Figure 1: we implicitly assumed that the three supports were perfectly aligned, but in reality, that is rarely the case. Suppose point 2 sits slightly too low relative to the line connecting points 1 and 3, meaning it makes no contact. The beam will deform somewhat before point 2 begins to carry any load. Consequently, the reaction forces will differ from those calculated using the flexible-beam approach. Alternatively, the beam might undergo slight deformation during assembly while still in an unloaded state, introducing internal stresses. Once again, the resulting reaction forces will differ from the 'perfect' scenario. We also observed the impact of tolerances on statically indeterminate trusses: in these cases, too, they significantly influence the structure's actual strength.

In conclusion, these often-overlooked statically indeterminate structures have indeed taught us a thing or two. So, why not dust off those old textbooks? You might find them surprisingly useful; especially if FEM calculations are part of your daily work.
For more inspiration, scan the QR-code for the video [V1].

REFERENCES
[1] E. Tempelman, "NSFD I – Statica", *Constructeur*, vol. 58 (10), pp. 26-28, 2019; E.Tempelman, "NSFD V – De tirannie van… ingenieursbalken-theorie), *Constructeur*, vol. 59 (4/5), pp. 24-27, 2020.
[2] E. Tempelman, "NSFD VI – Ouderwets de 21ste eeuw in: spaceframes", *Constructeur*, vol. 59 (6/7), pp. 24-27, 2020.
VIDEO
[V1] "DREAM Hall video 02 statics", *https://youtu.be/jBggrXAA7wg*

Colloquium dedicated to Dutch doyen of determinate design

Wim van der Hoek (1924-2019) was the Dutch doyen of design engineering principles. Statically determinate design was one of his guiding principles. To honour his legacy, the Wim van der Award was established. Since 2006, the award is presented every year to the person with the best graduation project in the field of design in mechanical engineering at a Dutch or Belgian university of technology or university of applied sciences.
To celebrate the 20th anniversary of the award, at the Precision Fair, on Thursday 19 November 2026 in Den Bosch (NL), a colloquium will be held: "Wim van der Hoek's Design Principles – Applying and Extending his Legacy". See page 33 for the programme.