BEYOND INERTIAL MATCH

In high-dynamic mechatronic systems, the inertial match principle is a widespread concept used by engineers to assist with actuator and transmission selection. Inertial match describes the optimal transmission ratio between payload and actuator inertia for minimum peak current and minimum thermal dissipation, where the effective inertia of the payload due to the transmission ratio is equal to the inertia of the actuator. This article presents an extension of the inertial match principle, for applications where the payload is subject to a constant force (such as gravity) or friction. Unlike the classical inertial match case, two different optimal values are now found, one for minimum peak current and one for minimum thermal dissipation.

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Introduction

In the early development phase of a mechatronic system, it is valuable to define a range of possible actuators and transmissions for the specific application. Inertial match is a principle that can help here, by defining the transmission ratio for minimum peak force/torque and minimum thermal dissipation.

This article starts with the description of a simplified mechatronic system and the relevant physical relations. Secondly, a refresher of the classical inertial match is presented, followed by an extension of the inertial match principle for applications with a payload subject to a constant force. The optimal transmission ratio values for minimum peak current and minimum thermal dissipation are extensively discussed, followed by guidelines to assist with actuator selection. Finally, friction effects are included, followed by some concluding remarks.

Relevant physical relations

The force/torque that a Lorentz-type actuator (e.g., a voice-coil actuator or a DC motor) can deliver is linearly dependent on the current: $F = KI$ or $T = KI$, with F and T the actuator force and torque, respectively, K the motor constant in [N/A] or [Nm/A], and I the current, meaning that minimum current is equivalent to minimum force/torque.

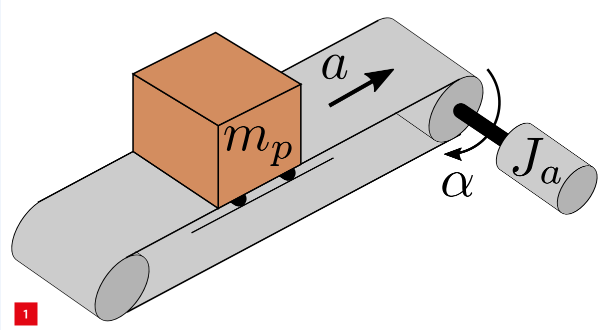
The maximum rms current applied to an actuator is limited by the allowable thermally dissipated energy W_{th} (= $\int RI(t)^2 \mathrm{d}t$), which is proportional to either $\int F(t)^2 \mathrm{d}t$ or $\int T(t)^2 \mathrm{d}t$, with R the electrical resistance of the actuator. The dissipated energy is proportional to the force/torque squared and the time the force/torque is acting.
The allowable thermal dissipation is determined by a combination of the maximum temperature of the actuator coil, the environment, the cooling method (air, water, etc.) and the mechanical design, which determines the thermal time constant that depends on the thermal resistance and thermal capacity of the actuator [1].

Maximum peak current of a power amplifier limits the peak force/torque an actuator can deliver. High peak currents in combination with other power amplifier requirements (noise level, linearity, dynamic properties, slew rate, etc.) can limit the availability of suitable power amplifiers significantly [2].

The transmission ratio between actuator and payload should be selected such that the peak current and thermal dissipation are minimised. Here, a simplified system is used to explain the inertial match concept and the effect of a payload subject to gravity on the inertial match criterion.

The example here consists of a payload mass (m_p = 1 kg) that is translated from one position to another with a rotational actuator, in this case a DC motor (J_a = 1,000 gcm²). The rotational motion is converted with pulleys and a belt to a translational motion, see Figure 1. The transmission ratio of this system is defined as:

$$i \equiv \frac{motion\ out}{motion\ in}\ [\mathrm{m/rad}] \quad (1)$$



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Simplified mechatronic system in which a rotary actuator with inertia J_a is accelerating a payload mass m_p via a rotary belt.

AUTHOR'S NOTE

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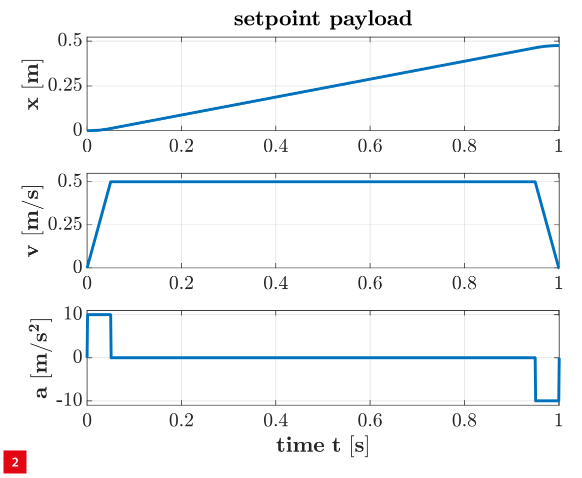

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Second-order setpoint for the simplified mechatronic system with displacement x, velocity v and acceleration a. Note that the acceleration/deceleration time here is small compared to the total setpoint time.

Note that the approach of this article also holds for motion systems with a transmission from translational to translational motion, or rotational to rotational motion (both with a dimensionless transmission ratio), and from translational to rotational motion (transmission ratio unit: [rad/m]). Rotational quantities (torque, moment of inertia, angular displacement, angular velocity, angular acceleration, etc.) can be replaced by translational quantities (force, mass, displacement, velocity, acceleration, etc.) and vice versa, depending on the mechatronic system. In this example, imperfections such as finite belt stiffness and play are neglected. Friction will be neglected at first and then discussed in a separate section.

Figure 2 shows the prescribed setpoint of the payload. The force required to move the payload according to Newton's Second Law is: $F_p = m_p a$. Here, F_p is the required force, m_p the payload mass and a the required payload acceleration. The force and resulting acceleration are delivered with the rotational actuator as: $F_p = T_p/i$ and $a = \alpha i$. Here, T_p is the actuator torque to move the payload and α the actuator angular acceleration to give the payload the desired acceleration. Combining the required payload force with the actuator relations results in: $T_p = m_p i^2 \alpha$. The moment of inertia of the payload as experienced at the actuator shaft is defined as: $J_p = m_p i^2$.

Classical inertial match

The classical inertial match criterion describes the optimal transmission ratio for minimising the torque required for acceleration. Imperfections are neglected, so torque is only required during acceleration. The minimum peak torque T in this case also corresponds to the minimum thermal dissipation, because only during acceleration energy is dissipated and this dissipation is proportional to T^2. The total moment of inertia (J_t) the actuator needs to accelerate, is the moment of inertia of the rotating part of the actuator itself (J_a), and the effective payload moment of inertia as described above: $J_t = J_a + m_p i^2$. The setpoint defines an acceleration a for the payload and, combined with the transmission ratio, this results in an angular acceleration of the actuator $\alpha = a/i$. The required torque is:

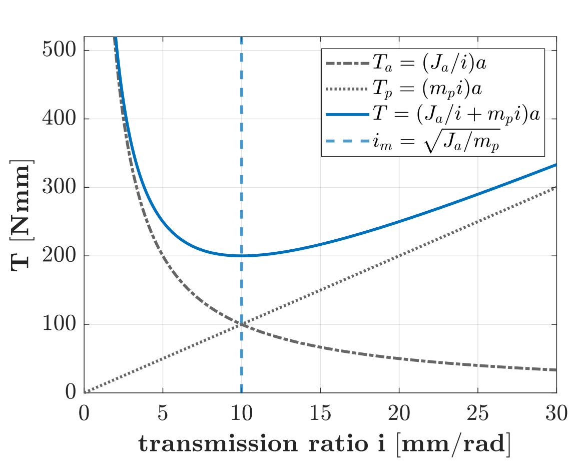
$$T = J_t \alpha = \left(J_a + m_p i^2\right) \frac{a}{i} = \left(\frac{J_a}{i} + m_p i\right) a \qquad (2)$$

To find the optimal inertial match transmission ratio i_m, the required torque is differentiated to i and the derivative is set equal to zero:

$$\frac{dT}{di} = 0 \rightarrow i_m = \sqrt{\frac{J_a}{m_p}} \qquad (3)$$

This optimal transmission ratio yields a minimum peak torque, resulting in minimum peak current and minimum thermal dissipation. Note that this does not minimise the mechanical power or kinetic energy delivered to the total system: a larger transmission ratio will always reduce the kinetic energy in the system by reducing the kinetic energy of the actuator, as the payload kinetic energy is determined by the setpoint.

Figure 3 shows the torque component required to accelerate the actuator moment of inertia T_a, the torque component to accelerate the payload T_p, and the total required torque T; see also the corresponding terms of Equation 2. When both terms are equal, the total required torque is minimum. In Figure 4, the setpoint of Figure 2 has been converted to actuator quantities for three different transmission ratios ($0.5i_m$, i_m and $1.5i_m$). The actuator torque is lowest for the transmission ratio with inertial match.



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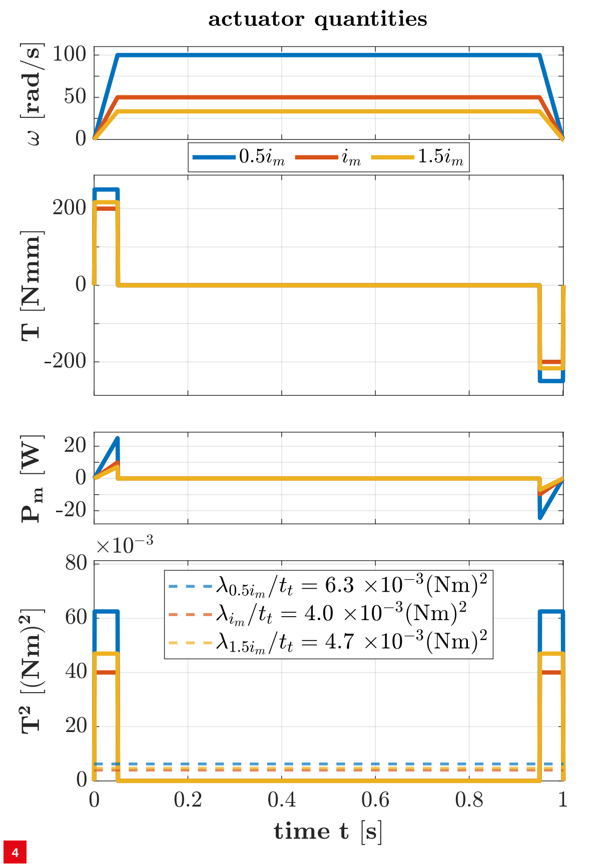
Torque terms to accelerate the actuator moment of inertia (T_a) and the payload mass (T_p), respectively, and the total torque (T), as a function of the transmission ratio i.



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Payload setpoint converted to actuator quantities for three transmission ratio values ($0.5i_m$, i_m and $1.5i_m$): angular velocity (ω), mechanical power (P_m), torque (T) and, in addition, torque squared (T^2, indication of thermal dissipation) with the average thermal dissipation indicator (λ/t_t) for three transmission ratio values included in the graph.

The average dissipated thermal power $\bar{P}_{th}$ over the total setpoint time t_t is also minimum in case of inertial match:

$$\bar{P}_{th} = \frac{1}{t_t}\int_0^{t_t} P_{th}\,dt = \frac{1}{t_t}R\int_0^{t_t} I^2\,dt \quad (4)$$
$$\left(= \frac{1}{t_t}\cdot\frac{R}{K^2}\int_0^{t_t} T^2\,dt\right)$$

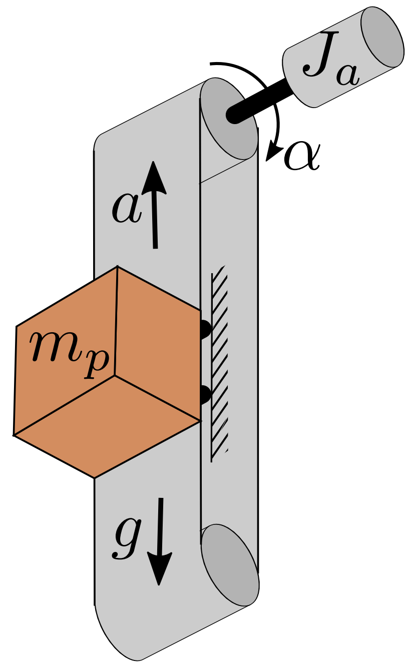
To improve readability, a thermal dissipation indicator λ is defined:

$$\lambda \equiv \int_0^{t_t} T^2\,dt \quad (5)$$

In this way: $\bar{P}_{th} \propto \frac{\lambda}{t_t}$. As expected, the higher transmission ratio ($1.5i_m$) leads to a lower mechanical power. Using power electronics with four-quadrant operation, the mechanical power during deceleration can be converted back to electrical power. In that case, the only consumed energy is the thermally dissipated energy [3].

Payload subject to a constant force

When the payload follows the setpoint while subject to a constant force such as gravity, there are two different optimal inertial match transmission ratio values found, namely an inertial match minimising the peak torque and



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Simplified mechatronic system rotated 90°. The payload in this situation is subject to gravity (while fixed to the belt).

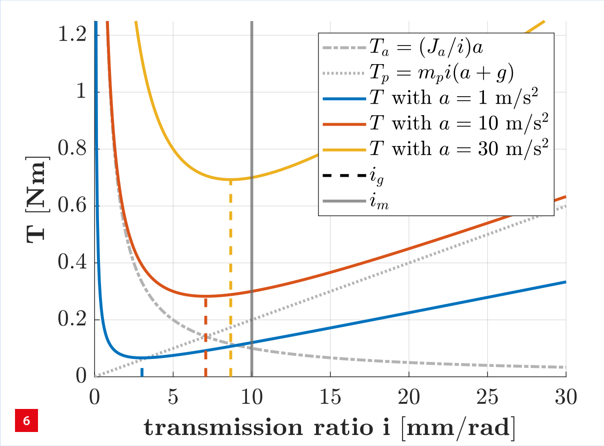
an inertial match minimising the thermal dissipation. The key difference is that the actuator must now continuously provide a torque to counteract gravity.

The movement as shown in the setpoint of Figure 2 is now assumed to be upwards in the 90°-rotated system (see Figure 5). The gravitational acceleration g yields an additional torque component for the actuator: $T = m_p g i$. For applications at a specific angle β with the horizontal, g can be replaced with $g\cdot\sin\beta$. Note that this constant force ($m_p g$) could also be replaced by any other constant load force that is independent of the motion direction, such as a constant preload force (but not a friction force!). In the following equations, g can also be replaced by any other constant force F_c divided by the payload mass ($g = F_c/m_{pl}$).

Minimum peak current

In this simplified system, the maximum torque (and maximum current) occurs during the initial acceleration phase, when the acceleration direction is opposite to the gravity direction. During deceleration the acceleration and gravity are in the same direction, which results in a lower required torque. The opposite holds when moving back downwards. The maximum torque required to accelerate the total moment of inertia and, additionally, to counteract the gravity of the payload is given by:

$$T = J_t\alpha + m_p g i = \left(J_a + m_p i^2\right)\frac{a}{i} + m_p g i = \frac{J_a}{i}a + m_p(a+g)i \quad (6)$$



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Required torque terms as a function of transmission ratio i to accelerate the actuator moment of inertia (T_a), accelerate the payload mass and compensate for gravity (T_p) for a = 10 m/s², respectively, and the total torque (T) for three values of the acceleration a.

As before, the actuator torque is differentiated to i and the derivative is set equal to zero to find the inertial match transmission ratio for minimum peak torque i_g:

$$\frac{dT}{di} = 0 \rightarrow i_g = \sqrt{\frac{a}{a+g} \cdot \frac{J_a}{m_p}} = \sqrt{\frac{a}{a+g}}\, i_m \tag{7}$$

The inertial match for minimum peak torque with the payload subject to gravity has an additional component to the original inertial match criterion, which depends on the maximum acceleration and gravitational acceleration. As shown in Figure 6, the inertial match transmission ratio is still balancing the actuator torque component (T_a) and the payload torque component (T_p), to find a minimum total required torque (T).

The difference with the classical inertial match is that the payload needs additional torque, to compensate for the gravity. For high accelerations the influence of gravity becomes negligible, but for accelerations in the order of magnitude of the gravitational acceleration, there is a significant difference between the inertial match without gravity incorporated and with gravity taken into account. As shown in Equation 7, this inertial match can also be seen as a correction factor multiplied with the classical inertial match transmission ratio i_m. For example, for an acceleration of 10 m/s² the correction factor is approximately $\sqrt{(1/2)} \approx 0.71$.

Minimum thermal dissipation

The inertial match criterion incorporating gravity that minimises thermal dissipation is calculated by analysing the total dissipated thermal energy during a movement. Here, the total move time needs to be considered, because now also an actuator torque is required during the constant-velocity phase to counteract the gravity at the payload. For minimum thermal energy dissipation, both the magnitude of the required torque and the relative time spent in various regions of the setpoint are of importance. The thermally dissipated energy for one movement is:

$$W_{th} \propto \lambda = \int_0^{t_t} T^2 dt = T_{acc}^2 t_{acc} + T_v^2 t_v + T_{dec}^2 t_{dec} \tag{8}$$

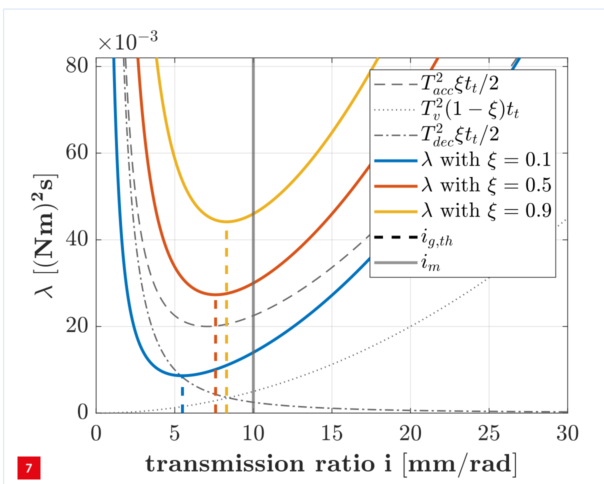
Here, T_{acc} and t_{acc}, T_v and t_v, and T_{dec} and t_{dec} are the required torque and time during acceleration, constant velocity and deceleration, respectively. By defining the motion duty cycle ξ as the ratio between the total acceleration and deceleration time and the total time ($\xi = (t_{acc} + t_{dec})/t_t$), and setting $t_{acc} = t_{dec}$ (symmetric setpoint), the thermal dissipation indicator is given by:

$$\lambda = \left(\frac{J_a}{i}a + m_p(g+a)i\right)^2 \frac{\xi t_t}{2} + \left(m_p g i\right)^2 (1-\xi) t_t + \left(-\frac{J_a}{i}a + m_p(g-a)i\right)^2 \frac{\xi t_t}{2} \tag{9}$$

Next, the thermally dissipated energy is differentiated to i and the derivative is set equal to zero to find the inertial match transmission ratio for minimum thermal dissipation:

$$\frac{d\lambda}{di} = 0 \rightarrow i_{g,th} = \sqrt[4]{\frac{\xi a^2}{\xi a^2 + g^2} \cdot \frac{J_a^2}{m_p^2}} = \sqrt[4]{\frac{\xi a^2}{\xi a^2 + g^2}}\, i_m \tag{10}$$

The optimal transmission ratio for minimum thermal dissipation is dependent on both the magnitude and fraction of acceleration time. For high acceleration values the effect of gravity is small. As dissipation is related to the torque squared, a high peak torque results in significant dissipation compared to the dissipation during the constant-velocity phase, even with a relatively small motion duty cycle ξ. When the accelerations are lower, the influence of the constant-velocity phase becomes more significant. For lower accelerations and smaller motion duty cycles, a balance is found between the dissipated energy during the acceleration, constant-velocity and deceleration phases.



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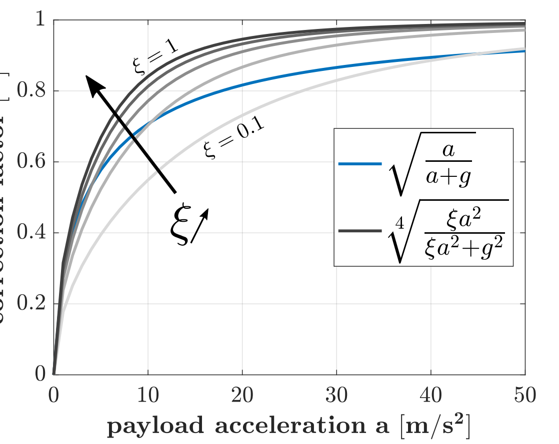
Thermal dissipation indicator as a function of transmission ratio i for the acceleration (acc), constant-velocity (v) and deceleration (dec) contributions for ξ = 0.5, and total thermal dissipation indicator for various motion duty cycle (ξ) values.



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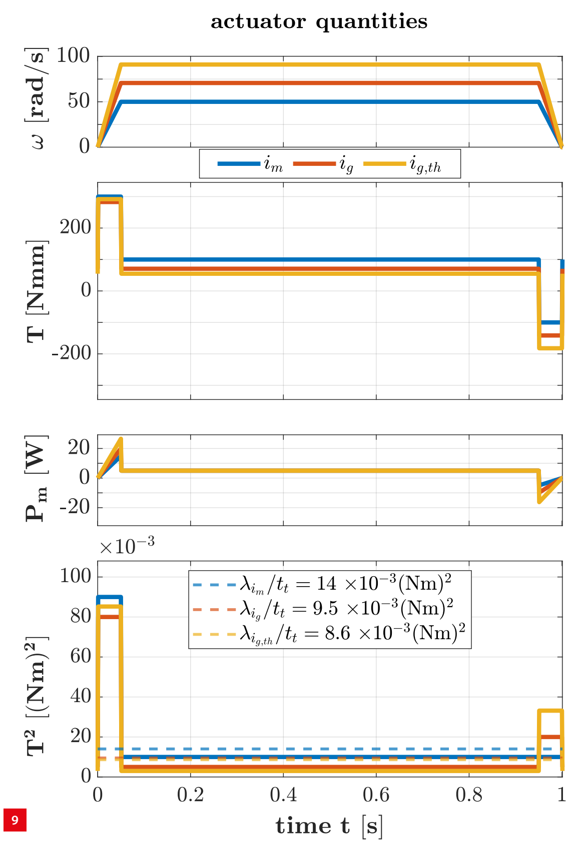
Correction factor for the classical inertial match transmission ratio as a function of acceleration (a): for minimum peak torque in blue, and for minimum thermal dissipation in grey, from light to dark with increasing motion duty cycle ξ.

In continuous processes, the motion duty cycle ξ is approximately 0 (e.g., pumps and ventilators with an approximately constant load), which yields an optimal transmission ratio for minimum thermal dissipation of 0 and requires infinitely high actuator velocities with zero force, which results in zero current and no dissipation (practically infeasible). Figure 7 shows the thermal dissipation indicator for different ξ values and transmission ratios (for the setpoint of Figure 2: $\xi = 0.1$), Equations 8 and 9 describe the corresponding thermal dissipation contributors.



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Actuator quantities for the simplified system with a payload subject to gravity, for the classical inertial match (i_m), the inertial match with minimum peak current (i_g), and the inertial match with minimum thermal dissipation ($i_{g,th}$). As in Figure 4, the average thermal dissipation indicator (λ/t_t) for the three transmission ratio values is included in the T^2 graph.

Figure 8 shows the correction factors for a mechatronic system with a payload subject to gravity. This correction factors need to be multiplied with the classical inertial match i_m. The correction factor for minimum peak torque is depicted in blue and the correction factor for minimum thermal dissipation with increasing motion duty cycle ξ is shown in light to dark grey. For accelerations up to about 50 m/s² a significant difference with the classical inertial match can be seen, while for higher accelerations the effect becomes less significant.

Figure 9 shows the actuator quantities for the simplified system of Figure 5, with the payload subject to gravity. As expected, the classical inertial match i_m does not provide the minimum peak torque or minimum thermal dissipation. The transmission ratio for minimum thermal dissipation $i_{g,th}$ shows the balance between dissipation during the acceleration/deceleration and the constant-velocity phase. The peak torque is a bit higher than the inertial match transmission ratio for minimum peak torque i_g, but the torque during constant velocity is smaller, resulting in a lower thermal dissipation.

Initial actuator selection

In the early phase of the development process of a mechatronic system, it is important to get a feel for the required actuator. The quoted mechanical power of an actuator is often used as a first estimate. Following the inertial matching principle as discussed above, the required mechanical power P is:

$$P = (J_a/i^2 + m_p)av \tag{11}$$

By filling in the different transmission ratio values derived above, different expressions for the initial power requirement estimate are derived. Using classical inertial match ($i_m = \sqrt{(J_a/m_p)}$) this results in:

$$P = 2m_p av = 2P_{p,m} \tag{12}$$

Here, $P_{p,m}$ is the maximum required power to move the payload. For a mechatronic system with a payload subject to gravity, the required maximum power for the payload is: $P_{p,g} = m_p(a + g)v$. Using the inertial match derived for minimum peak torque ($i_g = \sqrt{[a/(a + g)]}\cdot i_m$), the same relation as with classical inertial match can be found:

$$P = (J_a/i^2 + m_p)av + m_p gv = 2P_{p,g} \tag{13}$$

Note that for an acceleration of 10 m/s^2 the required power is double the power when the gravity is not taken into account: $P = 2P_{p,g} = 4P_{p,m}$. When minimising to thermal dissipation ($i_{g,th} = \sqrt[4]{\xi a^2/(\xi a^2 + g^2)} \cdot i_m$) with a payload subject to gravity, the total power is:

$$P = \sqrt{1 + \frac{g^2}{\xi a^2}}\, m_p a v + m_p (a + g) v = \sqrt{1 + \frac{g^2}{\xi a^2}}\, P_{p,m} + P_{p,g} \quad (14)$$

The mechanical power is definitely not the only quantity for actuator selection, but it certainly helps by defining a range of possible actuators, which supports the selection of the right actuator.

Including friction effects

Friction effects can be included at the payload in the simplified mechatronic system of Figure 5. In most mechatronic applications, the friction force F_μ (e.g., the friction force from preloaded bearings) is negligible compared to the acceleration forces, but in manufacturing systems high friction forces can be present (e.g., broaching). The friction force is implemented with a friction factor $k = F_\mu / m_{pl}$. Now, the same reasoning as before is followed.

The inertial match for minimum peak torque is given by:

$$i = \sqrt{\frac{a}{a + g + k}}\, i_m \quad (15)$$

The inertial match for minimum thermal dissipation is given by:

$$i = \sqrt[4]{\frac{\xi a^2}{\xi a^2 + g^2 + k^2 + 2\xi g k}}\, i_m \quad (16)$$

The inertial match for minimum thermal dissipation assumes a reciprocating (up- and downwards) motion, as the friction force is dependent of the motion direction, i.e. $F_\mu \propto \mathrm{sgn}(v)$ and works against the actuator during acceleration but helps during deceleration. The upwards and downwards motions individually result in different optimal transmission ratios, while combining both movements results in the global optimal transmission ratio for minimum thermal dissipation.

Independent of the application, inertial match does not automatically imply that the thermal dissipation is minimum. For example, for elevators (which are not high-dynamic applications) it is more interesting to use a balance mass at the actuator side with a transmission ratio $i = m_a / m_p$ to counteract the constant gravity force, or another significant change in the mechatronic system (e.g., an air spring). This will result in a lower total thermal dissipation without changing the transmission ratio to inertial match.

Concluding remarks

In this article, the inertial match criterion for high-dynamic mechatronic systems has been explained, in combination with an extension of this criterion when the payload is subject to a constant force. The classical inertial match defines a transmission ratio for minimum peak current (and force/torque) and minimum thermal actuator dissipation.

For the case of a payload subject to a constant force (gravity in this article), two different inertial matches have been found: one for minimum peak current and one for minimum thermal dissipation. For motion profiles with accelerations in the order of magnitude of the gravitational acceleration (approximately below 50 m/s^2) and sufficient constant-velocity time (motion duty cycle approximately below 0.3), the inertial match transmission ratio changes significantly.

A correction factor has been determined for correcting the classical inertial match, and a solution with friction forces has been derived. Dependent on the mechatronic application and the foreseen limiting factor, a selection between the two different inertial match criteria can be made.

In practice, inertial match is often used to estimate the required actuator power, not as a set-in-stone rule but more as a kind of guideline. For the case of a payload subject to gravity, the above has shown that classical inertial match underestimates the required mechanical power. Applying inertial match does not guarantee minimum thermal dissipation, but given an actuator type and a specific mechatronic system it yields the minimum thermal dissipation. Drastic changes to the mechatronic system can result in lower thermal dissipation, for example the addition of a gravity compensator (such as a low-stiffness mechanism or counter-masses), or the use of a different actuator type.

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